Cohomology

Def. A cochain complex C consists of ab groups C^n, $n \in \mathbb{Z}$ and homomorphisms $d^n: C^n \longrightarrow C^{n+1}$ such that $d^{n+1} \circ d^n = 0 \quad \forall n \in \mathbb{Z}$.

Def. The cohomology groups of a cochain complex C are

$$H^n C := \frac{\ker(d^n: C^n \to C^{n+1})}{\operatorname{im}(d^{n-1}: C^{n-1} \to C^n)}$$

Def. A cochain map $f: C \to D$ consists of homomorphisms $f^n: C^n \longrightarrow D^n$ such that $d^n \circ f^n = f^{n+1} \circ d^n \quad \forall n \in \mathbb{Z}$.

Such an f induces a homomorphism of cohomology groups:

$$H^n f: \quad H^n C \longrightarrow H^n D$$
$$[x] \longmapsto [f^n(x)]$$

Def. A cochain homotopy between cochain maps $f, g: C \to D$ consists of homomorphisms $s^n: C^n \longrightarrow D^{n-1}$ s.t. $d^{n-1} s^n + s^{n+1} d^n = f^n - g^n$.

Def. We define a functor $(\ )^\#$: (cochain cxs.) $\longrightarrow$ (chain cxs.) by $(C^\#)_n := C^{-n}$,

$$\begin{array}{ccccc} d^\#_n : & C^\#_n & \longrightarrow & C^\#_{n-1} \\ \| & \| & & \| \\ d^{-n} : & C^{-n} & \longrightarrow & C^{1-n} \end{array}$$

Analogously $(\ )_\#$: (ch cxs) $\to$ (coch. cxs.)
This is an iso of

(cochain homotopy)$^\#$ $\longrightarrow$ chain homotopy

and $H^n(C) = H_{-n}(C^\#)$

This means that the results of homological algebra can be applied for cohomology as well.

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This means that the results of homological algebra can be applied for cohomology as well.

For example: a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \to H^n A \xrightarrow{H^n f} H^n B \xrightarrow{H^n g} H^n C \xrightarrow{\partial} H^{n+1} A \to \cdots$$

This implies that cochain homotopic cochain maps induce the same map in cohomology.

Construction

C chain complex, A abelian gp.

We define a cochain complex $\underline{\mathrm{Hom}(C, A)}$ by

$$\mathrm{Hom}(C, A)^n = \mathrm{Hom}_{Ab}(C_n, A)$$

with pointwise addition

$$\begin{array}{ccc} \mathrm{Hom}(C,A)^n & \xrightarrow{d^n} & \mathrm{Hom}(C,A)^{n+1} \\ \| & & \| \\ \mathrm{Hom}_{Ab}(C_n, A) & & \mathrm{Hom}_{Ab}(C_{n+1}, A) \\ (f: C_n \to A) & \longmapsto & (f \circ d_{n+1}: C_{n+1} \to A) \end{array}$$

What is $H^n(\mathrm{Hom}(C, A))$? (Later: another UCT.)

__Def.__ X space, A an abelian group. The __singular cochain complex__ of X with coeffs in A is $\mathrm{Hom}(C_*(S(X); \mathbb{Z}), A)$.

The __chomology groups of X__ with coefficients in A are

$$\underline{H^n(X; A)} := H^n\Big(\mathrm{Hom}(C_*(S(X); \mathbb{Z}), A)\Big)$$

__Rmk.__ Cohomology can be defined for arbitrary simplicial sets Y as

$$H^n(Y; A) = H^n(\mathrm{Hom}(C_*(Y, \mathbb{Z}), A))$$

The complex $\mathrm{Hom}(C_*(Y; \mathbb{Z}), A)$ has a different reinterpretation:

Define a cochain cx $\underline{C^*(Y; A)}$ as

$$C^n(Y; A) = \mathrm{map}(Y_n, A) \quad \text{with pointwise addition}$$

Topology 1, lecture 24–25

$$d^n : C^n(Y;A) \longrightarrow C^{n+1}(Y;A) \qquad \text{coboundary}$$

$$(f: Y_n \to A) \longmapsto d^n(f): Y_{n+1} \longrightarrow A$$

$$(d^n(f))(\underbrace{y}_{\in Y_{n+1}}) = \sum_{i=0}^{n+1} (-1)^i f(\underbrace{d_i^*(y)}_{\in Y_n})$$

$d^{n+1} \circ d^n = 0$. These alternate definitions agree.

<u>Prop.</u> For any s.set Y and ab group A the following homomorphism

$$\begin{array}{ccc} \mathrm{Hom}(C_n(Y;\mathbb{Z}), A) & \xrightarrow[\text{evaluate on generators}]{\cong} & \mathrm{map}(Y_n, A) \\ \| & & \\ \mathrm{Hom}(\mathbb{Z}[Y_n], A) & & \end{array}$$

defines an iso of cochain complexes:

$$\mathrm{Hom}(C_*(Y;\mathbb{Z}), A) \cong C^*(Y;A).$$

Pf. (sketch) Univ. prop. of free ab groups,

$$\mathrm{Hom}(\mathbb{Z}[S], A) \xrightarrow[\text{eval. on generators}]{\cong} \mathrm{map}(S, A) \quad \text{is an iso.}$$

Check the compat with the bdry. □

<u>Properties of cohomology groups</u>

1) <u>Homotopy invariance</u>: for $f, g: X \to Y$ homotopic cont. maps we have $H^n f = H^n g : H^n(X;A) \to H^n(Y;A)$

0) Cohomology is a <u>contravariant functor</u>:

$$(\mathrm{Top}) \xrightarrow{S} (\mathrm{sSets}) \xrightarrow{C_*(-,\mathbb{Z})} (\mathrm{ch.\ cxs}) \xrightarrow{\mathrm{Hom}(-,A)} (\mathrm{coch.\ cxs})^{op} \xrightarrow{H^n} (\mathrm{ab.\ gps.})^{op}$$

The proof of 1) is clear by this diagram: every step preserves homotopy.

the contravariance happens here, the directions remain the same at all other places.

2) <u>Long exact sequence</u>

For $Y \subseteq X$ we define the <u>relative cohomology</u>:

$$\underline{H^n(X,Y;A)} := H^n\left(\mathrm{Hom}\left(\frac{C_*(S(X);A)}{C_*(S(Y);A)}, A\right)\right)$$

$$0 \longrightarrow C_*(S(Y),\mathbb{Z}) \longrightarrow C_*(S(X);\mathbb{Z}) \longrightarrow \frac{C_*(S(X);\mathbb{Z})}{C_*(S(Y);\mathbb{Z})} \longrightarrow 0 \qquad (*)$$

is a sh. ex. sq. of chain cxs. Take $\mathrm{Hom}(-,A)$:

$$0 \longleftarrow \mathrm{Hom}(C_*(S(Y);\mathbb{Z}),A) \longleftarrow \mathrm{Hom}(C_*(S(X);\mathbb{Z});A) \longleftarrow \mathrm{Hom}\left(\frac{C_*(S(X);\mathbb{Z})}{C_*(S(Y);\mathbb{Z})},A\right) \longleftarrow 0$$

$\frac{C_*(S(X);\mathbb{Z})}{C_*(S(Y);\mathbb{Z})}$ is dimensionwise free ab $\Rightarrow$ $(*)$ splits dimensionwise

$\Rightarrow$ Hom preserves exactness (this is not true generally)

So we get a l.e.s.:

$$\cdots \longrightarrow H^n(X,Y;A) \longrightarrow H^n(X;A) \xrightarrow{incl^*} H^n(Y;A) \xrightarrow{\partial} H^{n+1}(X,Y;A) \longrightarrow \cdots$$

3) <u>Excision</u>: $B \subseteq Y \subseteq X$ nested top spaces, $\overline{B} \subseteq \mathring{Y}$.

Then the inclusion $(X\setminus B, Y\setminus B) \hookrightarrow (X,Y)$ induces an isomorphism

$$H^n(X,Y;A) \xrightarrow{\cong} H^n(X\setminus B, Y\setminus B, A).$$

(Note that the natural map goes in the opposite direction than for homology.)

In the proof of excision for homology we have seen that 24.1.2018 (C.W.)

the map $\frac{C_*(S(X\setminus B))}{C_*(S(Y\setminus B))} \longrightarrow \frac{C_*(X)}{C_*(Y)}$ is a quasi-isomorphism,

i.e. an isomorphism on homology.

In general, $\mathrm{Hom}(-,A)$ does <u>not</u> preserve quasi-isomorphisms.

e.g. $C := (\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z})$ is acyclic, i.e. $H_*C = 0$

but $\mathrm{Hom}(C,\mathbb{Z}/2\mathbb{Z}) = (\cdots \longleftarrow 0 \longleftarrow \mathbb{Z}/2\mathbb{Z} \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{\cong} \mathbb{Z}/2\mathbb{Z})$,

$$H^2(\mathrm{Hom}(C,\mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z} \neq 0.$$

Prop. Let $f: C \to D$ be a map between levelwise projective complexes. Then if f is a quasi-isomorphism then it is also a chain homotopy equivalence, i.e. $\exists g: D \to C$, $f \circ g \simeq id_D$, $g \circ f \simeq id_C$.

Assuming the Prop., it follows that

$$\mathrm{Hom}\left(\frac{C_*(S(X))}{C_*(S(Y))}; A\right) \longrightarrow \mathrm{Hom}\left(\frac{C_*(S(X \setminus B))}{C_*(S(Y \setminus B))}; A\right)$$

is a quasi-iso and hence

$$H^n(X, Y; A) \xrightarrow{\sim} H^n(X \setminus B, Y \setminus B; A)$$

is an iso $\forall n \geq 0$.

Lemma. (Fund. lemma of homological algebra)

Let $P = (\cdots \to P_n \to P_{n-1} \to \cdots \to P_0)$ be a levelwise projective complex,
$Q = (\cdots \to Q_n \to Q_{n-1} \to \cdots \to Q_0)$ an acyclic inaugmented complex,
i.e. $\cdots \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \twoheadrightarrow H_0 Q$ is exact.

Then the following hold:

(1) Any map $f: H_0 P \to H_0 Q$ is induced by a map $f: P \to Q$ of complexes.

(2) Any two maps $f, f': P \to Q$ with $H_0 f = H_0 f'$ are chain homotopic.

Pf:

$$\begin{tikzcd} P_0 \arrow[r, dashed, "f_0"] \arrow[d] & Q_0 \arrow[d, "d_0"] \\ H_0 P \arrow[r, "f"] & H_0 Q \end{tikzcd}$$

P_0 projective, d_0 surjective $\Rightarrow \exists f_0$ lift.

Induction: suppose we already have constructed a partial chain map $f: P \to Q$ up to degree n.

$$\begin{tikzcd}
P_{n+1} \arrow[r, dashed, "f_{n+1}"] \arrow[d, "d_{n+1}"'] & Q_{n+1} \arrow[d, "d_{n+1}"] \\
P_n \arrow[r, "f_n"'] & Q_n \arrow[d, "d_n"] \\
& {}
\end{tikzcd}$$

$d f_n d = d d f_n = 0$

$\Rightarrow$ the composite $f_n d$ has img

This proves existence (1).

Let $f, f' : P \longrightarrow Q$ be chain maps, $H_0 f = H_0 f' = f$.

Want to construct a homotopy $h : P_* \longrightarrow Q_{*+1}$, $f - f' = hd + dh$.

$$\begin{tikzcd}
& Q_1 \arrow[d] \\
P_0 \arrow[ur, dashed, "h_0"] \arrow[r, "f_0 - f_0'"'] & Q_0 \arrow[d, "d_0"] \\
& H_0 Q
\end{tikzcd}$$

Induction: supp. a homotopy is constructed up to degree n.

$$\begin{tikzcd}
& Q_{n+2} \arrow[d, "d_{n+1}"] \\
P_{n+1} \arrow[ur, dashed, "h_{n+2}"] \arrow[r, "f_{n+1} - f'_{n+1} - h_n d"'] & Q_{n+1} \arrow[d, "d_{n+1}"] \\
& {}
\end{tikzcd}$$

$$d\,(f - f' - hd) = (f - f')\,d - dhd$$
$$= (hd + dh)\,d - dhd = hdd = 0.$$

$\square$

<u>Cor.</u> (Special case of the Prop.)

Let C be an acyclic and levelwise projective complex.

Then C is chain contractible, i.e. $\exists$ homotopy $h : C_* \longrightarrow C_{*+1}$,

$$1 = dh + hd.$$

Pf: Both id_C and 0 induce the zero map in $H_0 C = 0$. $\square$

To deduce the general case, we need the mapping cone of a chain map. $f : C \longrightarrow D$.

<u>Def.</u> $\underline{C(f)_n} := D_n \oplus C_{n-1}$, $\underline{d^{C(f)}}$: $C(f)_n \xrightarrow{\begin{pmatrix} d^D & f \\ 0 & -d^C \end{pmatrix}} C(f)_{n-1}$

This sits in the short exact sequence:

$$0 \longrightarrow D \xrightarrow{i} C(f) \xrightarrow{p} C[1] \longrightarrow 0$$

with the shifted complex $C[1]_n = C_{n-1}$, $d^{C[1]} = d^C$.

Under the identification $H_{n+1}(C[1]) = H_n C$, the connecting homomorphism $\partial: H_{n+1} C[1] \longrightarrow H_n D$ takes the form $f_*: H_n C \longrightarrow H_n D$.

$c \in C_{n-1}$ cycle $\overset{p}{\Rightarrow}$ $(0,c) \in C(f)_n \overset{\text{dif.}}{\to} (f(c), 0) \in C(f)_{n-1} \Rightarrow f(c) \in D_n$.

We obtain the long exact sequence

$$\cdots \longrightarrow H_n C \xrightarrow{f} H_n D \longrightarrow H_n C(f) \xrightarrow{p_*} H_{n-1} C \longrightarrow \cdots$$

From this it follows that $C(f)$ is acyclic $\Leftrightarrow$ f is a quasi-iso.

PF OF PROP: C, D levelwise projective, f quasi-iso.

Want to construct $g: D \to C$, $g \circ f \simeq id_C$, $f \circ g \simeq id_D$.

$C(f)$ is again levelwise projective and hence contractible by the Cor., so there is $h: C(f)_* \to C(f)_{*+1}$, $1_{C(f)} = hd + dh$.

Define $g: D \to C$ by the composite

$$g_n: D_n \xrightarrow{i} D_n \oplus C_{n-1} \xrightarrow{h} D_{n+1} \oplus C_n \xrightarrow{p} C_n.$$

This is a chain map:

$$(g_n \circ d)(x) = p\,h\,(dx, 0) = p\,h\,d\,(x,0)$$
$$= p\,(1 - dh)(x,0)$$
$$= -p\,dh(x,0)$$
$$= d\,p\,h\,(x,0) = (d\,g_{n+1})(x).$$

Chain homotopy equivalence: define a homotopy $h^C: C_* \to C_{*+1}$ as the composite map

$$C_n \xrightarrow{j} D_{n+1} \oplus C_n \xrightarrow{h} D_{n+2} \oplus C_{n+1} \xrightarrow{p} C_{n+1}$$

Then
$$(g_n \circ f_n)(c) = ph\,(fc, 0) = ph\left(d^{C(f)}(0,c) + (0, dc)\right) =$$
$$= p\,(1 - dh)(0,c) + \underbrace{ph\,(0, dc)}$$
$$= c - p\,dh\,(0,c) + h^C dc$$
$$= c + dph\,(0,c) + h^C dc$$
$$= c + dh^C(c) + h^C dc.$$

This shows one direction of the htp equivalence; the other direction is similar. $\square$

For $m \geq 1$:

$$H^m(S^n; A) \cong \begin{cases} A & n = m \text{ or } 0 \\ 0 & \text{otherwise} \end{cases} \qquad H^m(D^n, \partial D^n; A) \cong \begin{cases} A & n = m \text{ or } 0 \\ 0 & \text{otherwise} \end{cases}$$

These can be deduced in the same way as for homology.

Cellular cohomology

X CW-complex, $C_n^{cell}(X; A) := H_n(X_n, X_{n-1}; A)$ cellular chain complex

with differential $H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})$

Def. Cellular cochain complex $C^*_{cell}(X; A) := \mathrm{Hom}(C_*^{cell}(X; \mathbb{Z}), A)$

Prop. $H^n(X; A) \cong H^n(C^*_{cell}(X; A)) \quad \forall n \geq 0$

Pf: As for homology. $\square$

Relation between homology and cohomology.

Ex. $C_*^{cell}(\mathbb{C}P^n) = (\cdots \to 0 \to \overset{2n}{\mathbb{Z}} \to 0 \to \overset{2n-2}{\mathbb{Z}} \to \cdots \to 0 \to \mathbb{Z})$

$\Rightarrow C^*_{cell}(\mathbb{C}P^n) = (\cdots \leftarrow 0 \leftarrow A \leftarrow 0 \leftarrow A \leftarrow \cdots \leftarrow 0 \leftarrow A)$

$$\Rightarrow H^m(\mathbb{C}P^n; A) \cong \begin{cases} A & m = 2n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

So like for the sphere, $H^m(\mathbb{C}P^n; A) \cong H_m(\mathbb{C}P^n; A)$.

This should not be expected in general: (co)homology is co(ntra)variant

$H^m(X; A) \not\cong \mathrm{Hom}(H_m(X), A)$, e.g. $X = \mathbb{R}P^n$.

But for a chain complex C and ab. group A define

$$\Phi : H^n(\mathrm{Hom}(C, A)) \longrightarrow \mathrm{Hom}(H_n C, A)$$
$$[f : C_n \to A] \longmapsto ([c] \mapsto f(c))$$

Well-definedness: • $[c] = [c'] \Rightarrow c = c' + dc'' \Rightarrow f(c) = f(c') + f(dc'')$

$$= f(c') + \underbrace{d^*f}_{0}(c'')$$

• $[f] = [f'] \Rightarrow f = f' + g \circ d \Rightarrow \forall c \in Z_n C: \; f(c) = f'(c) + (g \circ d)(c)$

$$= f'(c) + g(\underset{\sim}{dc}).$$

<u>Prop.</u> Φ has an additive section (if C is levelwise projective)

$$\sigma : \operatorname{Hom}(H_n C, A) \longrightarrow H^n(\operatorname{Hom}(C, A))$$

In particular, Φ is surjective.

PF:
$$0 \longrightarrow Z_n \longrightarrow K_n \longrightarrow \underset{\text{boundaries}}{\overset{\subseteq C_{n-1}}{B_{n-1}}} \longrightarrow 0$$

B_{n-1} is free as a subgroup of C_{n-1}

$\Rightarrow$ the sequence splits $\forall n$

Define σ as follows: for a map $f : H_n C \longrightarrow A$ let

$\sigma(f) : C_n \longrightarrow A$ be the composite $C_n \xrightarrow{r} Z_n C \longrightarrow H_n C \xrightarrow{f} A$.

Check well-definedness.

σ is a section: $(\Phi \circ \sigma)([f : C_n \longrightarrow A])(\overset{H_n C}{\overset{\in}{[c]}})$

$$= \Phi(C_n \longrightarrow Z_n C \longrightarrow H_n C \longrightarrow A)([c])$$

$$= \text{evaluate}\ (C_n \xrightarrow{r} Z_n C \longrightarrow H_n C \longrightarrow A)([c])$$

$$= [r \underbrace{c}_{\in Z_n}] = [c] \qquad \Rightarrow \quad \Phi \circ \sigma = \mathrm{id}.$$

□

<u>Note.</u> M. Palmer: Morse theory Hauptseminar SS 2018

29.1.2018

Universal coefficient theorem for cohomology

For a chain complex C we have a natural homomorphism

$$\Phi : H^n(\operatorname{Hom}(C, A)) \longrightarrow \operatorname{Hom}(H_n C, A)$$
$$[f : C_n \longrightarrow A] \longmapsto \{[x] \mapsto f(x)\}$$

This is a split epimorphism.

"I expected that all of you in Franke's class will say 'Oh, we know what it is.' But even Franke can't do everything". (About Ext)

Recall the Fundamental lemma of homological algebra:

R ring, M,N left R-modules, $P_* \to M$, $Q_* \to N$ <u>projective resolutions</u>

i.e. $\cdots \to P_n \xrightarrow{d} P_{n-1} \to \cdots \to P_0 \xrightarrow{\varepsilon} M \to 0$ is exact and all P_i are projective.

For an R-module P TFAE.

1) For every epimorphism of R-modules $f: M \twoheadrightarrow N$ and every homomorphism $g: P \to N$ there is a homomorphism $\tilde{g}: P \to M$ s.t. $f \circ \tilde{g} = g$:

$$\begin{tikzcd} & M \arrow[d, "f"] \\ P \arrow[ur, dashed, "\tilde{g}"] \arrow[r, "g"'] & N \end{tikzcd}$$

2) For every epimorphism of R-modules $f: M \twoheadrightarrow N$ the natural homomorphism $\mathrm{Hom}_R(P,f): \mathrm{Hom}_R(P,M) \to \mathrm{Hom}_R(P,N)$ is surjective.

3) P is a direct summand of a free R-module.

P is <u>projective</u> if 1—3) hold.

<u>Rmk.</u> Every module has a proj. resolution:

Inductive construction : $P_{-2} := 0$, $P_{-1} := M$, $d_{-1} = 0 : M \to 0$.

$n \geq 0$: choose a free module P_n and an R-linear surjection $P_n \twoheadrightarrow \ker(P_{n-1} \xrightarrow{d_{n-1}} P_{n-2})$, and then define $d_n : P_n \to P_{n-1}$ as the composite $P_n \twoheadrightarrow \ker(P_{n-1} \xrightarrow{d_{n-1}} P_{n-2}) \xhookrightarrow{\text{incl}} P_{n-1}$

$f: M \to N$ a hom. of R-modules. Then there is a chain map $\tilde{f}: P_* \to Q_*$ over f unique up to chain homotopy over f.

$$\begin{tikzcd} \vdots \arrow[d] & \partial & \vdots \arrow[d] \\ P_1 \arrow[rr, "\tilde{f}_1"'] \arrow[d] & \partial & Q_1 \arrow[d] \\ P_0 \arrow[rr, "\tilde{f}_0"'] \arrow[d] & \partial & Q_0 \arrow[d] \\ M \arrow[rr, "f"'] & & N \end{tikzcd}$$

<u>Cor.</u> ($M=N$, $f = \mathrm{id}$) Let $P_* \to M$ and $Q_* \to M$ be two projective resolutions of M. Then there is a chain homotopy $\tilde{f}: P_* \to Q_*$ over M unique up to chain homotopy.

Pf: Let $\tilde{f}: P_* \to Q_*$ be a chain map over id_M and $\tilde{g}: Q_* \to P_*$ a chain map over id_M.

Then $\tilde{g} \circ \tilde{f}: P_* \to P_*$ is a chain self-map over id_M.

Since any two chain maps over the same homomorphism are chain homotopic, $\tilde{g} \circ \tilde{f} \simeq id_{P_*}$. Similarly $\tilde{f} \circ \tilde{g} \simeq id_{Q_*}$ □

Def. R a ring, M, N left R-modules. Choose a proj. resolution $P_* \to M$ and define $\underline{Ext^n_R(M,N)} := H^n(Hom(P_*, N))$

Rmk. $Ext^n_R(M,N)$ is independent of the choice of proj. resolution up to preferred isomorphisms: (i.e. the iso is not chosen)

Let $Q_* \to M$ be another proj resolution. Then there is a chain htp equivalence $\tilde{f}: P_* \to Q_*$ over id_M unique up to cochain htp. This induces a cochain homotopy equivalence

$$Hom_R(\tilde{f}, N): Hom_R(Q_*, N) \to Hom_R(P_*, N)$$

unique up to cochain homotopy.

This induces an iso in cohomology:

$$H^n(Hom_R(\tilde{f}, N)): H^n(Hom_R(Q_*, N)) \xrightarrow{\cong} H^n(Hom_R(P_*, N)) = Ext^n_R(M,N)$$

Rmk. $Ext^n_R(M,N)$ is functorial in both variables.

Let $g: N \to N'$ be a homomorphism of R-modules

$\to$ $Hom_R(P_*, g): Hom_R(P_*, N) \to Hom_R(P_*, N')$

$\Rightarrow$
$$\begin{array}{ccc} H^n(Hom_R(P_*, g)): & H^n(Hom_R(P_*, N)) \longrightarrow & H^n(Hom_R(P_*, N')) \\ \| & \| & \| \\ \underline{f_*} = Ext^n_R(M, g): & Ext^n_R(M, N) \longrightarrow & Ext^n_R(M, N') \end{array}$$

$\Rightarrow$ covariant functoriality in N.

Let $f: M' \to M$ be a hom of R-modules.

Let $P_* \to M$, $P'_* \to M'$ be proj resolutions chosen for the defs of the Ext groups.

The fund. lemma provides a chain map $\tilde{f}: P_* \to P'_*$ over f.

$$\Rightarrow \quad \mathrm{Hom}_R(\tilde{f}, N): \quad \mathrm{Hom}_R(P'_*, N) \longrightarrow \mathrm{Hom}_R(P_*, N)$$

$$\Rightarrow \quad H^n(\mathrm{Hom}_R(\tilde{f}, N)): \quad H^n(\mathrm{Hom}_R(P'_*, N)) \longrightarrow H^n(\mathrm{Hom}_R(P, N))$$

$$\underline{f^*} = \mathrm{Ext}^n_R(\tilde{f}, N): \quad \mathrm{Ext}^n_R(M', N) \longrightarrow \mathrm{Ext}^n_R(M, N)$$

It is easily seen that $f^* \circ g_* = g_* \circ f^*$.

__Prop.__ $\mathrm{Ext}^0_R(M,N)$ is naturally isomorphic to $\mathrm{Hom}_R(M,N)$.

Pf: Let $P_* \to M$ be the chosen projective resolution.

$$\begin{aligned}
\mathrm{Ext}^0_R &= H^0(\mathrm{Hom}_R(P_*, N)) \\
&= \ker\left(\mathrm{Hom}_R(P_0, N) \xrightarrow{\mathrm{Hom}_R(d_1, N)} \mathrm{Hom}_R(P_1, N)\right) \\
&\cong \mathrm{Hom}_R(\mathrm{coker}(d_1: P_1 \to P_0), N) \\
&\underset{\text{via } d_0}{\cong} \mathrm{Hom}_R(M, N)
\end{aligned}$$

(there is nothing lower so there is no im to factor with)

by the univ prop of the quotient

$$\begin{array}{c} P_1 \\ \downarrow d_1 \\ P_0 \\ \downarrow d_0 \\ M \\ \downarrow \\ 0 \end{array}$$

__Def.__ An __extension of M by N__ is a short exact sequence of R-modules:

$$0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$$

Two extensions are __equivalent__ if we have a comm. diag. of R-mod. maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow \cong & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Now we can talk about equivalence classes of extensions.

Topology I, Lecture 26

There is a natural isomorphism:

$$\mathrm{Ext}^1_R(M,N) \cong (\text{extensions of } M \text{ by } N)/\text{equivalence}$$

$$\underbrace{[\tilde{f}_1 : P_1 \to N]}_{\substack{\cap \\ H^1(\mathrm{Hom}_R(P_*, N))}} \longmapsfrom \begin{array}{ccc} \downarrow & & \\ P_2 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ P_1 & \overset{\tilde{f}_1}{\dashrightarrow} & N \\ \downarrow & & \downarrow \\ P_0 & \overset{\tilde{f}_0}{\dashrightarrow} & E \\ \downarrow & & \downarrow \\ M & = & M \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

According to Schwede, it is a lot of fun to show that this is actually an isomorphism.

Properties: • $\mathrm{Ext}^n_R(-,-)$ preserves direct sums in both variables.

• a short exact sequence of R-modules

$$0 \to M \to N \to T \to 0$$

gives rise to a long exact sequence

$$0 \to \mathrm{Hom}_R(T,S) \to \mathrm{Hom}_R(N,S) \to \mathrm{Hom}_R(M,S) \to$$
$$\to \mathrm{Ext}^1_R(T,S) \to \mathrm{Ext}^1_R(N,S) \to \mathrm{Ext}^1_R(M,S) \to \quad \cdots$$

and similarly in the other variable.

Ex. R a field (a skew field would be enough, commutativity is not essential)
$\Rightarrow$ every R-module is free (every vector space has a basis).

We choose $0 \to 0 \to M \xrightarrow{=} M \to 0$

as the projective resolution (we resolve M by itself).

$$\Rightarrow H^n(\mathrm{Hom}_R(P_*, N)) = 0 \quad \forall n \geq 0$$
$$\| $$
$$\mathrm{Ext}^n_R(M,N)$$

(We didn't actually use the full power of being a field, we only needed this specific proj. resolution.)

Ex. $R = \mathbb{Z}$. $\quad$ R-modules = abelian groups

Subgroups of free ab groups are free (hence projective)

$$0 \longrightarrow 0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

free ab groups

$$\Rightarrow \quad \mathrm{Ext}^n_R(M,N) = 0 \quad \forall n \geq 2.$$

This is why it is common to write $\underline{\mathrm{Ext}(M,N)} := \mathrm{Ext}^1(M,N)$.

Ex. Free resolution of $\mathbb{Z}/n\mathbb{Z}$:

$$\underbrace{\cdots \to 0 \to 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}}_{P_*} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

$\mathrm{Hom}_{\mathbb{Z}}(P_*, \mathbb{Z})$:

$$\begin{tikzcd} 0 & \mathrm{Hom}(\mathbb{Z},N) \arrow[l] \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\mathbb{Z},N) \arrow[l] \arrow[d, phantom, "\cong"] & 0 \arrow[l] \\ & N & N \arrow[l, "\cdot n"'] & \end{tikzcd}$$

$$\Rightarrow \mathrm{Ext}\left(\mathbb{Z}/n\mathbb{Z}, N\right) = N/nN$$

Ex. $R := \mathbb{Z}/4\mathbb{Z}$ $\quad$ (this is a "homologically complicated" ring)

$M := \mathbb{Z}/2\mathbb{Z}$ as an R-module

Projective resolution:

$$\underbrace{\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z}}_{P_*} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

(We have to go on infinitely to the left since $\mathbb{Z}/4\mathbb{Z}$ is not projective.)

$$\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}\left(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\right) = H^n\left(\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(P_*, \mathbb{Z}/2\mathbb{Z})\right)$$

$$= H^n\left(\cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \leftarrow \cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \leftarrow 0\right)$$

$$\xleftarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xleftarrow{\cdot 2} \cdots \xleftarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z}$$

$$= \mathbb{Z}/2\mathbb{Z} \quad \forall n \geq 0$$

The "homologically complicatedness" means that all the Ext groups are nontrivial (even though R seems to be a relatively "simple" ring).

Now we finally turn to the (2^{nd}) UCT.

<u>Thm.</u> (UCT) C chain complex, $\forall n \in \mathbb{Z}$: C_n is free abelian.

Let A be an abelian group. Then the kernel of the split epimorphism

$$\Phi: H^n(\mathrm{Hom}(C, A)) \longrightarrow \mathrm{Hom}(H^n C, A)$$

$$[f: C_n \to A] \longmapsto ([x] \mapsto f(x))$$

is naturally isomorphic to $\mathrm{Ext}(H_{n-1}C, A)$.

<u>Cor.</u> ($C = C_*(S(X), \mathbb{Z})$, X a space) There is a nat. sh. ex. sq.

$$0 \to \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}), A) \to H^n(X;A) \xrightarrow{\Phi} \mathrm{Hom}(H_n(X;\mathbb{Z}), A) \to 0$$

PF OF UCT: A homomorphism $f: C_n \to A$ represents a class in the kernel of Φ iff $f(Z_n) = 0$ where $Z_n = \{x \in C_n \mid dx = 0\}$.

Then for $f \circ d = 0$:

$$\ker(\Phi) = \frac{\{f: C_n \to A \mid f(Z_n) = 0\}}{\{g \circ d: C_n \to A \mid g \in \mathrm{Hom}(C_{n-1}, A)\}}$$

The short exact sequence

$$0 \to Z_n \xrightarrow{\text{incl}} C_n \xrightarrow{d_n} \underset{\mathrm{im}(d_n)}{B_{n-1}} \to 0$$

.. .. with this group.

$$\cong \frac{\mathrm{Hom}(B_{n-1}, A)}{\{f \in \mathrm{Hom}(B_{n-1}, A) \mid f \text{ admits an additive extension to } C_{n-1}\}}$$

The short exact sequence

$$0 \to Z_{n-1} \xrightarrow{\text{incl}} C_{n-1} \underset{\sigma}{\overset{d_{n-1}}{\rightleftarrows}} B_{n-2} \to 0$$

.. of free abelian groups. $\Rightarrow$ Split and $Z_{n-1} \oplus \sigma(B_{n-2}) = C_{n-1}$

$$\begin{array}{c} B_{n-1} \subseteq Z_{n-1} \subseteq C_{n-1} \\ \downarrow \\ A \end{array}$$

So $f: B_{n-1} \to A$ admits an additive extension to C_{n-1} iff f admits an additive extension to Z_{n-1}.

In formulas:

$$\ker \Phi \cong \frac{\operatorname{Hom}(B_{n-1}, A)}{\operatorname{im}\left(\operatorname{Hom}(Z_{n-1}, A) \longrightarrow \operatorname{Hom}(B_{n-1}, A)\right)} =$$

The short exact sequence $\underbrace{0 \longrightarrow B_{n-1} \longrightarrow Z_{n-1}}_{P_*} \longrightarrow H_{n-1}C \longrightarrow 0$ is a projective resolution of the ab. group $H_{n-1}C$.

$$= H^1(\operatorname{Hom}(P_*, A)) = \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}C, A) = \operatorname{Ext}(H_{n-1}C, A)$$ □

<u>Ex.</u> If $H_n(X;\mathbb{Z})$ is free abelian $\forall n \geq 0$:

then $\operatorname{Ext}(H_{n-1}X, A) = 0$ and

$$\Phi : H^n(X;A) \longrightarrow \operatorname{Hom}(H_n(X,\mathbb{Z}); A)$$

is an iso $\forall n \geq 0$.

Recall:
$$H_n(\mathbb{R}P^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

$$\Rightarrow \quad H^0(\mathbb{R}P^2; A) \xrightarrow{\cong} \operatorname{Hom}(H_0(\mathbb{R}P^2;\mathbb{Z}), A) \cong A$$

$$H^1(\mathbb{R}P^1; A) \xrightarrow{\cong} \operatorname{Hom}(H_1(\mathbb{R}P^1;\mathbb{Z}), A) \cong \{x \in A \mid 2x = 0\} \quad \text{2-torsion}$$

$$0 \to \operatorname{Ext}(H_1(\mathbb{R}P^2;\mathbb{Z}), A) \to H^2(\mathbb{R}P^2; A) \xrightarrow{\cong} \operatorname{Hom}(\underbrace{H_2(\mathbb{R}P^2;\mathbb{Z})}_{0}, A) \longrightarrow 0$$

$$\operatorname{Hom}(H_2(\mathbb{R}P^2;\mathbb{Z}), A) = 0$$

$$\operatorname{Ext}(H_1(\mathbb{R}P^2;\mathbb{Z}), A) \cong \operatorname{Ext}(\mathbb{Z}/2\mathbb{Z}, A) \cong A/2A$$

Alternate calculation: use the cellular cochain complex:

$$H^n(\operatorname{Hom}(C^{\text{cell}}_*(X;\mathbb{Z}), A)) \cong H^n(X; A)$$

for all CW-complexes X. The argument is the same as for homology, just with arrows reversed.

Recommended: compute the cohomology of $\mathbb{R}P^2$ this way as well.

Exam. Feb 20, 9:00 – 10:30, GHS + KHS (we may actually fit into the GHS), arrive before 8:50

Everything from the lectures and the exercises can be used w/o proof

No electronic devices may be used. Smartphones have to be deposited.

Studentenausweis + ID with picture needed.

Questions can be answered in English or German.

Definition, reproduce easy proof, calculation, proof of a statement not proved in the lecture.

The exams will be corrected by Feb 21.

—— o ——

Cup product on cohomology

This is something that does not have an analogue in homology.

Thm. Let C be a chain complex of free ab. groups, A abelian group. Then there is a split s.e.s.

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}C, A) \longrightarrow H^n(\operatorname{Hom}(C, A)) \longrightarrow \operatorname{Hom}(H_nC, A) \longrightarrow 0$$

$$[f: C_n \longrightarrow A] \longmapsto ([x] \longmapsto f(x))$$

Special case. X space, $C = C_*(S(X); \mathbb{Z})$

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X; \mathbb{Z}), A) \longrightarrow H^n(X, A) \longrightarrow \operatorname{Hom}(H_n(X; \mathbb{Z}), A) \longrightarrow 0$$

Def. C, D chain complexes. We define their tensor product $C \otimes D$ by

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j, \quad d_n^{C\otimes D}(x \otimes y) = d_i^C(x) \otimes y + (-1)^i x \otimes d_j^D(y)$$

where $x \in C_i$, $y \in D_j$, $i+j=n$

Rmk. $d(d(x \otimes y)) = d(d(x) \otimes y + (-1)^i x \otimes d(y))$

$$= \underbrace{d^2(x)}_{0} \otimes y + \underbrace{(-1)^{i-1} dx \otimes dy + (-1)^i dx \otimes dy}_{0} + (-1)^{2i} x \otimes \underbrace{d^2 y}_{0} = 0$$

The same definition can be given for cochain complexes (just put the indices into the superscript.)

Rmk. X, Y CW-complexes and $X \times Y$ has the product CW-structure, then

$$C_*(X \times Y; \mathbb{Z}) \cong C_*(X; \mathbb{Z}) \otimes C_*(Y; \mathbb{Z}). \quad \text{(PO)}$$

Construction. X simplicial set. The cup product / Alexander–Whitney map is the chain map (for a ring R)

$$\underline{\cup}: \quad C^*(X; R) \otimes_{\mathbb{Z}} C^*(X; R) \longrightarrow C^*(X; R)$$

defined as follows: $\forall f \in C^m(X; R), \forall g \in C^n(X; R): \forall x \in X_{m+n}$:

$$\underline{(f \cup g)}(x) := f(d^*_{front}\, x) \cdot g(d^*_{back}\, x)$$

where $\underline{d_{front}}: [m] \longrightarrow [m+n], \quad d_{front}(i) = i$

$\underline{d_{back}}: [n] \longrightarrow [m+n], \quad d_{back}(i) = m+i$

(these are both monotone injective maps, their only common value is m)

We have $\cup(f, g) = f \cup g \in C^{m+n}(X; R)$.

Thm. 1) The Alexander–Whitney map is a chain map, i.e.

$$d(f \cup g) = d(f) \cup g + (-1)^m f \cup d(g).$$

2) For every morphism of s.sets $\alpha: X \longrightarrow Y$, $f \in C^m(Y; R)$, $g \in C^n(Y; R)$:

$$\alpha^*(f) \cup \alpha^*(g) = \alpha^*(f \cup g).$$

3) The cup product is associative and unital, i.e. if $h \in C^p(X; R)$ then

$$(f \cup g) \cup h = f \cup (g \cup h) \quad \text{and} \quad 1 \cup f = f \cup 1 = f$$

for $1 \in C^0(X; R)$ the constant function with value 1.

Pf: 1) For $d_{front}: [m] \longrightarrow [m+n]$ we have (by def.)

$$d_i \circ d_{front} = \begin{cases} d_{front} \circ d_i & 0 \leq i \leq m+1 \\ d_{front} & m+1 \leq i \leq n+m+1 \end{cases}$$

Similarly for $d_{back}: [n+1] \longrightarrow [m+n+1]$ we have

$$d_i \circ d_{back} = \begin{cases} d_{back} & 0 \leq i \leq n \\ d_{back} \circ d_{i-n} & n \leq i \leq n+m+1 \end{cases}$$

$$d(f\cup g) = (df)\cup g + (-1)^m f\cup(dg) \in C^{m+n+1}(X;R) \qquad f\in C^m(X;R),\ g\in C^n(X;R)$$

Let $x\in X_{m+n+1}$.

$$\begin{aligned}
(d(f\cup g))(x) &= \sum_{i=0}^{m+n+1}(-1)^i (f\cup g)(d_i^*(x))\\
&= \sum_{i=0}^{m+n+1}(-1)^i f\left(d^*_{front}\left(d_i^*(x)\right)\right)\cdot g\left(d^*_{back}\left(d_i^*(x)\right)\right)\\
&= \sum_{i=0}^{m}(-1)^i f\left(d^*_{front}\left(d_i^*(x)\right)\right)\cdot g\left(d^*_{back}\left(d_i^*(x)\right)\right) +\\
&\quad + \sum_{i=1}^{n+1}(-1)^{i+m} f\left(d^*_{front}\left(d^*_{i+m}(x)\right)\right)\cdot g\left(d^*_{back}\left(d^*_{i+m}(x)\right)\right)\\
&= \sum_{i=0}^{m+1}(-1)^i f\left(d_i^*\left(d^*_{front}(x)\right)\right)\cdot \underbrace{g\left(d^*_{back}(x)\right)}_{\text{indep. of } i} +\\
&\quad + \sum_{i=0}^{n+1}(-1)^{m+i} \underbrace{f\left(d^*_{front}(x)\right)}_{\text{indep. of } i}\cdot g\left(d_i^*\left(d^*_{back}(x)\right)\right)\\
&= (df)\left(d^*_{front}(x)\right)\cdot g\left(d^*_{back}(x)\right) + f\left(d^*_{front}(x)\right)\cdot(dg)\left(d^*_{back}(x)\right)(-1)^m\\
&= ((df)\cup g)(x) + (-1)^m (f\cup(dg))(x)
\end{aligned}$$

one extra term in both sums, cancel out

$$f\left(d^*_{m+1}\left(d^*_{front}(x)\right)\right)\cdot g\left(d^*_{back}(x)\right) = f\left(d^*_{front}(x)\right) g\left(d^*_{back}(x)\right) = f\left(d^*_{front}(x)\right)\cdot g\left(d^*_0\left(d^*_{back}(x)\right)\right)$$

2) $\alpha: X\to Y$, $f\in C^m(X;R)$, $g\in C^n(Y;R)$, $x\in X_{m+n}$

$$\begin{aligned}
(\alpha^*(f\cup g))(x) &= (f\cup g)(\alpha_{m+n}(x)) = f\left(d^*_{front}\,\alpha_{m+n}(x)\right)\cdot g\left(d^*_{back}\,\alpha_{m+n}(x)\right) =\\
&= f\left(\alpha_m\, d^*_{front}(x)\right)\cdot g\left(\alpha_n\, d^*_{back}(x)\right) =\\
&= (\alpha^* f)\left(d^*_{front}\,x\right)\cdot(\alpha^* g)\left(d^*_{back}\,x\right) = (\alpha^* f\cup\alpha^* g)(x).
\end{aligned}$$

α is a simplicial map

3) $f\in C^m(X;R)$, $g\in C^n(X;R)$, $h\in C^p(X;R)$

$$\begin{tikzcd}
{[n]} \arrow[dr, bend left, "d_{front}"] & \\
{[m]} \arrow[r, "d_{middle}", "i\mapsto m+i"'] & {[m+n+p]} \\
{[p]} \arrow[ur, bend right, "d_{back}"'] &
\end{tikzcd}$$

$$((f \cup g) \cup h)(x) = \Big(f(d^*_{front}(x)) \cdot g(d^*_{middle}(x))\Big) \cdot h(d^*_{back}(x))$$

$\|$ associativity in R

$$(f \cup (g \cup h))(x) = \big(f(d^*_{front}(x))\big) \cdot \Big(g(d^*_{middle}(x)) \cdot h(d^*_{back}(x))\Big)$$

For $f \in C^m(X; R)$, $1 \in C^0(X; R)$, $x \in X_m$:

$$(f \cup 1)(x) = f(d^*_{front}(x)) \cdot 1(d^*_{back}(x)) = f(x) \cdot 1(x) = f(x)$$

<u>Def.</u> A <u>differential graded ring</u> consists of:

- a cochain complex C (could be done for homology too)
- biadditive maps $\cdot : C^m \times C^n \longrightarrow C^{m+n}$ that are associative and unital and satisfy the Leibniz rule: $d(xy) = d(x)\, y + (-1)^m x\, d(y)$

<u>Examples.</u> 1) X simplicial set, $C^*(X; R)$ is a DGR under $\cup$

2) de Rham complex of a smooth manifold M is a DGR under $\wedge$

de Rham's thm.: $\underset{\wedge}{H^*_{deRham}(M)} \cong \underset{\cup}{H^*_{sing}(M; \mathbb{R})}$ as graded rings

<u>Construction.</u> If C is a DGR, its cohomology inherits a product:

if $x \in C^m$, $y \in C^n$ are cocycles then we set $[x] \cdot [y] := [x \cdot y]$.

This is well-defined: for $z \in C^{m-1}$, $[x] = [x + dz]$

$$(x + dz) \cdot y = xy + (dz)y = xy + d(zy)$$

$$\Rightarrow [(x + dz)\, y] = [xy + d(zy)] = [xy]. \quad \checkmark$$

$\Rightarrow$ $\cdot : H^m C \times H^n C \longrightarrow H^{m+n} C$ makes cohomology into a graded ring.

<u>Ex.</u> X space, $H^*(X; \mathbb{R}) = H^*(C^*(S(X); \mathbb{R}))$ is a graded ring under $\cup$.

<u>Prop.</u> $f : X \longrightarrow Y$ continuous map of spaces. Then the induced map $f^* : H^*(Y; R) \longrightarrow H^*(X; R)$ is a homomorphism of graded rings. $\square$

<u>Thm.</u> Let R be a commutative ring. Then the cup product of $H^*(X;R)$ is graded-commutative: $\forall x \in H^m(X;R)$ $\forall y \in H^n(X;R)$: $x \cup y = (-1)^{m\cdot n} y \cup x$.
(Koszul sign rule)

<u>Rmk.</u> $d(xy) = d(x)y + (-1)^m x \cdot d(y)$ has the same sign convention.

<u>Rmk.</u> In general, $x \cup y \neq (-1)^{mn} y \cup x$ on the chain level, i.e. for $x \in C^m(X;R)$, $y \in C^n(X;R)$. The formula holds only after passing to cohomology.

PF OF THM: We define the $\cup_1$-product:

<u>$\cup_1$:</u> $C^m(X;R) \otimes C^n(X;R) \longrightarrow C^{m+n-1}(X;R)$, $\quad x \in X_{m+n-1}$:

$$(f \cup_1 g)(x) = \sum_{i=0}^{m-1} (-1)^{(m-i)(n+1)} f\left((d_i^{out})^*(x)\right) \cdot g\left((d_i^{inn})^*(x)\right)$$

where $d_i^{out}: X_m \to X_{m+n+1}$, $d_i^{inn}: X_n \to X_{m+n+1}$ are the injective monotone maps with imgs:

$$im(d_i^{out}) = \{0, \dots, i\} \cup \{i+n, \dots, m+n-1\}$$
$$im(d_i^{inn}) = \{i, \dots, i+n\}$$

Then we have (after tedious calculations, for masochists only):

$$d(f \cup_1 g) = (df) \cup_1 g + (-1)^m f \cup_1 (dg) - (-1)^{m+n} f \cup g - (-1)^{(m+1)(n+1)} g \cup f$$

Spec. case: $df = 0$ and $dg = 0$

$$\Rightarrow \quad d(f \cup_1 g) = -(-1)^{m+n}(f \cup g) - (-1)^{(m+1)(n+1)} g \cup f$$

$$\Rightarrow \quad (-1)^{m+n} d(f \cup_1 g) = f \cup g - (-1)^{mn} g \cup f$$

$$\Rightarrow \quad 0 = [f \cup g] - (-1)^{mn} [g \cup f]$$

$$\Rightarrow \quad [f] \cup [g] = (-1)^{mn} [g] \cup [f].$$

□